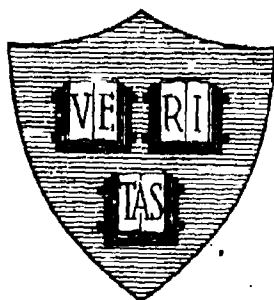


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NOTES ON DECISION AND CONTROL



May 1972

Technical Report No. 633



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ON THE EQUIVALENCE OF INFORMATION STRUCTURES IN STATIC AND DYNAMIC TEAMS

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In team theory, we visualize N decision makers (DM) or team members each receiving individual information. There are two important classes of information structures: (i) each DM's information only depends on certain primitive random variables ξ of the external world (ii) each DM's information depends not only on ξ but also the actions taken by other DMs. The first class can be called STATIC team theory or static information structure and the second DYNAMIC in the sense that the time order in which the DMs act is irrelevant in the first case while causality plays a definite role in the second. DM who acts later obviously cannot affect the information received by DM who acts precedent or concurrent with him. Radner [1] has established certain necessary, and sufficient conditions for optimality of the decisions of the DMs for a wide class of static teams. Ho and Chu [2] solved the linear-quadratic-gaussian dynamic team problems by showing their equivalence to the static case under certain further assumptions on the information structure of the team. In this note we generalize the conditions of equivalence to non-linear and nongaussian cases.

Let ξ be a random vector defined on a basic probability space $(\Omega, \mathcal{F}, \mathcal{P})$. $u = [u_1, u_2, \dots, u_N]$ be the decision (control) vector of the N DMs. Let z_i be the information available to the i th DM,

$$z_i = \eta_i(\xi, u_j, u_k \text{ etc.}), = \eta_i(\xi, \underline{u}_j) \quad (1)$$

where η_i are assumed to be measurable functions of ξ and \underline{u}_j etc. In general this information is affected by the decision variables of some of the DMs that act before i . Let j be one of those DMs which do so, we say DM j is a precedent of DM i and denote it by $j \prec i$. In (1) \underline{u}_j represents the decisions of all those j such that $j \prec i$. For causal system, there is always some starting DMs without any precedents for them and their information depends only on ξ . The precedence relation is assumed to be fixed in advance and does not depend on random events and decision outcomes. This precedence relation is generally a partial ordering [2]. We can then partition the N DMs into the following disjoint sets:

$$\mathcal{N} = \{1, 2, \dots, N\} = \bigcup_{l=1}^k \mathcal{N}_l,$$

where

$$\mathcal{N}_1 = \{i \mid z_i = \eta_i(\xi)\}, \text{ the set of starting DMs,}$$

$$\mathcal{N}_2 = \{i \mid z_i = \eta_i(\xi, \underline{u}_j), j \in \mathcal{N}_1\}$$

$$\vdots$$

$$\mathcal{N}_k = \{i \mid z_i = \eta_i(\xi, \underline{u}_j), j \in \bigcup_{l=1}^{k-1} \mathcal{N}_l, i \notin \bigcup_{l=1}^{k-1} \mathcal{N}_l\}$$

The decision rule (control law) of DM i is assumed to be a function of its own information

$$u_i = \gamma_i(z_i) \quad i = 1, \dots, N \quad (2)$$

where γ_i belongs to the set Γ_i of all admissible (Borel) measurable functions from the space of z_i to the space of u_i . Consequently for fixed $\gamma = [\gamma_1, \dots, \gamma_N]$, z_i induces subfields $\mathcal{F}_i \subset \mathcal{F}$. The above set up is in complete harmony with the usual dynamic stochastic system statement of a team problem. For example, we have,

$$x_{t+1} = f(x_t, u_t, w_t) \quad t = 1, \dots, N \quad (3)$$

$$y_t = h(x_t, v_t) \quad (4)$$

where x_1, w_t, v_t are external random variables playing the part of ξ and $\{y_\tau \mid \tau \in \text{subset of } \{1, \dots, t\}\} = z_t$ are the information available to u_t . The point is that the intermediate variables x_t are irrelevant in a decentralized setting and can always be eliminated via substitution. Only u and ξ and the information structure as specified by (1) count.

Definition 1. Information z_j is said to be nested in z_i if there exists a measurable function f_{ij} from the space of z_i to the space of z_j such that

$$z_j = f_{ij}(z_i) \quad \forall \xi \text{ and } u \quad (5)$$

Equivalently, we say z_j is nested in z_i if

$$\mathcal{F}_j \subset \mathcal{F}_i \quad \forall \gamma \in \Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_N \quad (6)$$

Definition 2. An information structure $\eta = [\eta_1, \dots, \eta_N]$ is said to be partially nested if for all $j, i \in \{1, \dots, N\}$, $j \neq i$ implies z_j is nested in z_i .

By definition, in static team there are no precedents for any of the decision makers; hence, any static team is trivially partially-nested.

A particularly simple and common form of partially nested information is

$$z_i = \begin{bmatrix} \{z_j | j \rightarrow i\} \\ z_i' = \eta_i'(\xi, \underline{u}_j) \end{bmatrix} \quad \forall i \quad (7)$$

In other words if \underline{u}_j affects the information received by the i th DM, then the i th DM will also know what the j th DM knows (as part of z_i). In this case, f_{ij} is simply a row matrix of $\{1, 0\}$ which picks out certain components of z_i .

Definition 3. Information z_i is said to be equivalent to information z_j if z_i and z_j are mutually nested, i. e. f_{ij} in (5) is invertible. The equivalent z_i and z_j referred here may represent the information of two different DMs or two different forms of information available to one DM. In the latter case, any strategy $u_i = \gamma_i(z_i)$ that can be realized by z_i can always be duplicated by z_j and vice versa. Yet the computation of strategies based on one information structure may be simpler than the other particularly if the latter is of static nature.

In the theorem below we shall state a condition on the equivalence between dynamic and static information structure. To this ends, suppose we can write

$$\begin{aligned} z_i &\triangleq \eta_i(\xi, \underline{u}_j) \\ &= \eta_i(\hat{\eta}_i(\xi), \underline{u}_j) \triangleq K_i(\hat{\eta}_i(\xi); \underline{u}_j) \quad \text{for fixed } \underline{u}_j \end{aligned} \quad (8)$$

Theorem 1. Given a team with partially nested information structure η , if furthermore (8) is true and K_i is invertible for all i and \underline{u}_j , then the information structure $\hat{z}_i \triangleq \hat{\eta}_i(\xi)$ is equivalent to $z_i = \eta_i(\xi, \underline{u}_j)$ for $i = 1, 2, \dots, N$.

Proof: It is clear that we can choose

$$z_i = \eta_i(\xi) = \hat{\eta}_i(\xi) = \hat{z}_i \quad i \in N_1 \text{ (i. e. } K_i = I, \text{ the identity function).}$$

Next we write from (8)

$$z_i = \eta_i(\hat{\eta}_i(\xi), \underline{u}_j) = \eta_i(\hat{\eta}_i(\xi), \gamma_j(\hat{z}_j)) \quad \begin{matrix} j \in N_1 \\ i \in N_2 \end{matrix}$$

Since $\hat{z}_j = z_j$ is partially nested, $\gamma_j(\hat{z}_j)$ is a known quantity. Hence, we can write

$$\hat{z}_i = \hat{\eta}_i(\xi) = K_i^{-1}(z_i; \gamma_j(z_j)), \quad \begin{matrix} i \in \mathcal{N}_2 \\ j \in \mathcal{N}_1 \end{matrix}$$

and

$$z_i = K_i(\hat{z}_i; \gamma_j(\hat{z}_j)),$$

This process can obviously be iterated for $i \in \mathcal{N}_3, \mathcal{N}_4, \dots, \mathcal{N}_k$. The collected equivalent information structure $\hat{\eta} = [\hat{\eta}_1, \dots, \hat{\eta}_N]$ is a static one.

Examples. In all examples below we assume (7) and illustrate particular useful cases of (8).

$$(i) \quad z_i = \begin{bmatrix} \{z_j | j \neq i\} \\ z_i^1 = H_i \xi + \sum_{j \neq i} D_{ij} u_j \end{bmatrix} \iff \begin{bmatrix} \{z_j | j \neq i\} \\ \hat{z}_i^1 \triangleq z_i^1 - \sum_{j \neq i} D_{ij} \gamma_j(z_j) = H_i \xi \end{bmatrix}$$

The above transformation can be used recursively to produce a set of \hat{z}_i which are linear combinations of ξ only. This trick was used to prove the optimality of a wide variety of LQG problems with partially nested information structure [2].

$$(ii) \quad z_i = \left[\begin{array}{l} \{z_j | j \neq i\} \\ z_i' = h_i(\xi) + d_i(u_j) \end{array} \right] \longleftrightarrow \left[\begin{array}{l} \{z_j' | j \neq i\} \\ \hat{z}_i' = z_i' - d_i(\gamma_j(z_j)) = h_i(\xi) \end{array} \right]$$

Example (ii) shows that one of the crucial requirements for equivalence is that the effect of u be additive.

$$(iii) \quad z_i = \left[\begin{array}{l} \{z_j | j \neq i\} \\ z_i' = A_i(u_j)h_i(\xi) + b_i(u_j) \end{array} \right]$$

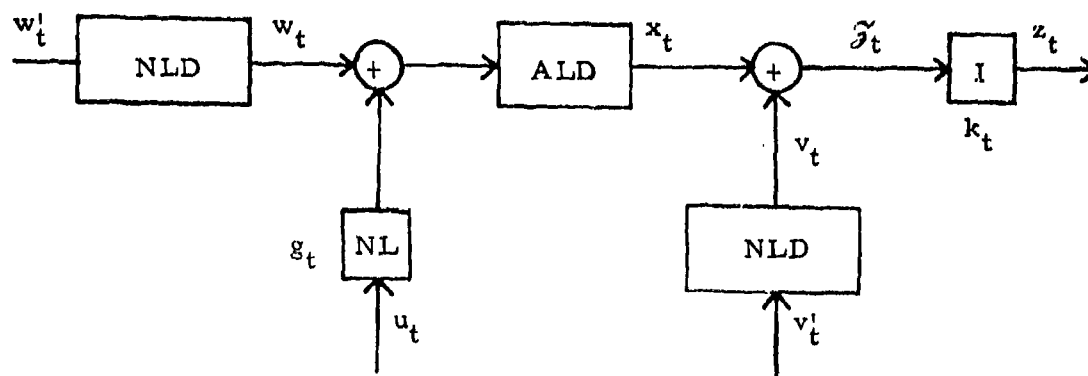
$A_i(u_j)$ is an invertible matrix for all admissible u , then again

$$z_i \longleftrightarrow \left[\begin{array}{l} \{z_j' | j \neq i\} \\ \hat{z}_i' = A_i^{-1}(\gamma_j(z_j)) [z_i' - b_i(\gamma_j(z_j))] = h_i(\xi) \end{array} \right]$$

The example shows that if the effect of the previous u is multiplicative then invertibility in some form is required. For example, if u_j turns off some measurement system such that $A_i(u_j) = 0$ for all $i \neq j$ then the system may still be partially nested but not equivalent to a static one.

Examples (i) - (iii) can be combined to yield the following assertions:

Assertion 1. The dynamic team in Figure 1 has an equivalent static information structure



NLD - Nonlinear Dynamic System

NL - Nonlinear Transformation, g_t

ALD - Additive Linear Dynamic Systems, Eq. (*)

I - Invertible Transformation, k_t

$$x_{t+1} = g(u_t) + Ax_t + w_t \quad (*)$$

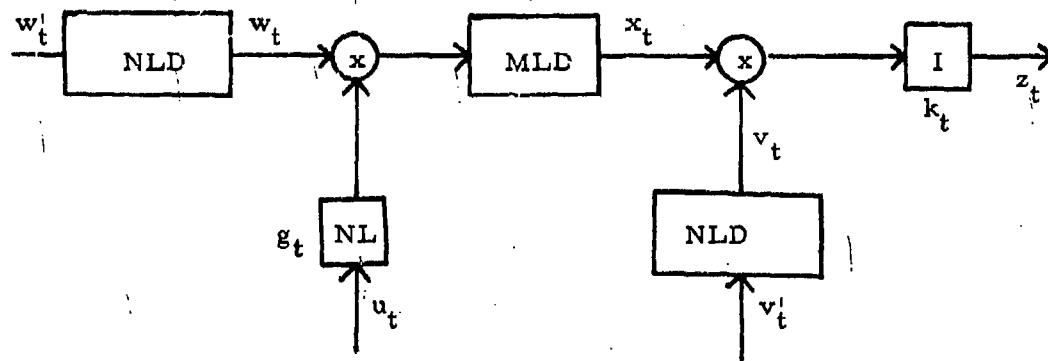
$$z_t = k_t(\tilde{z}_t) \quad \tilde{z}_t = \begin{bmatrix} Hx_t + v_t \triangleq z_t^1 \\ \{z_\tau^1 \mid \tau < t\} \end{bmatrix}$$

x_1, w_t^1, v_t^1 are externally specified random variables.

$$z_t \iff \begin{bmatrix} z_t^1 - H \sum_{i=1}^t A^{t-1} g_i(u_i) = \hat{z}_t^1 \\ \{\hat{z}_\tau^1 \mid \tau < t\} \end{bmatrix} \triangleq \hat{z}_t$$

Figure 1

Assertion 2. The dynamic team in Figure 2 has an equivalent static information structure provided $(g_t(u_t)A)^{-1}$ exist for any admissible u_t .



MLD - Multiplicative Linear Dynamic System, Eq. (†)

(x) - Multiplication of a vector by a scalar

$$x_{t+1} = g_t(u_t)A x_t w_t \quad (†)$$

$$z_t = k_t(\mathcal{Z}_t) \quad \mathcal{Z}_t = \begin{bmatrix} v_t x_t \triangleq z_t' \\ \{z_\tau' \mid \tau < t\} \end{bmatrix}$$

v_t, w_t are scalar random sequences externally specified x_1 is a random variable.

$$z_t \iff \begin{bmatrix} \prod_{i=1}^t (g_i(u_i)A)^{-1} z_t' = \hat{z}_t' \\ \{\hat{z}_\tau' \mid \tau < t\} \end{bmatrix}$$

Figure 2

Finally, we have the easy generalization of the Radner theorem.

Theorem II. Let a dynamic team have partially nested information structure η which is also equivalent to a static one $\hat{\eta}$. Let the performance function of the team be

$$J(\gamma) = E\{\mathcal{J}[u, \xi]\}$$

where ξ is a primitive random vector with given prior probability distribution $p(\xi)$. If (a) \mathcal{J} is convex and differentiable in u_i for all i and for a. e. ξ , (b) $J(\gamma)$ is locally finite* and (c) $\inf_{\gamma} J(\gamma) > -\infty$, then there is a unique local optimal decision $u_i = \gamma_i^*(z_i)$ for all i . Furthermore, if

$$J(\gamma) = E\left\{\frac{1}{2} u^T Q u + u^T S \xi + u^T c\right\} \quad (9)$$

where Q, S, c are constants, ξ is gaussian with finite covariance, $\hat{\eta}_i$ linear in ξ for all i , then γ_i^* is linear in \hat{z}_i and possibly nonlinear in z_i .

Also we have the extension

Corollary. If $p(\xi)$ in theorem II has the form

$$p(\xi) = f[(\xi - m)^T X^{-1} (\xi - m)] \quad (10)$$

then the optimal decision rule to (9) is again linear in \hat{z}_i .

Proof: Density functions of the form (10) are called spherically (elliptically) invariant, of which gaussian distribution is a special case. Vershik [3] and Blake and Thomas [4] have shown that all mean-square estimation problems with this class of random variables have linear solutions. More specifically, the Bayesian conditional mean $E[\xi | \hat{z} = H\xi]$ is linear in \hat{z} , and it is always true that

* For definition, see [1].

$$E(\xi | \hat{z}) = m + XH^T(HXH^T)^{-1}(\hat{z} - Hm), \quad (11)$$

when ξ is spherically invariant.

There is no loss of generality by letting $m = 0$. If we assume linear decision rule

$$u_i = A_i \hat{z}_i + b_i \quad (12)$$

and solve for the optimal A_i and b_i , i.e., substitute (11) and (12) into Eq. (19) of [2], we find, that

$$\sum_j Q_{ij} A_j (H_j X H_i^T) = - S_i X H_i^T \quad \forall i \quad (13)$$

$$b^T = c^T Q^{-1} \quad (14)$$

Everything remains the same as the gaussian case, except X here is not necessarily the covariance of ξ .

Conclusion

The matter discussed in this report concerns the reduction of information structures in dynamic team decision making to static cases. All the DMs are assumed to be rational. Partially-nested information structure is a system in which the DM knows what his precedents have known and by this he deduces what they have done. Based on this concept, the condition on information equivalence has been derived. The essence of information obtained may be disguised in various forms in many dynamic team problems. Our theorems clarify the information structure in several classes of problems.

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ON THE RELATIVE LEADERSHIP PROPERTY
OF STACKELBERG STRATEGIES

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ABSTRACT

The relative leadership property of Stackelberg strategies has been investigated via a scalar nonzero-sum two-person differential game problem. It is shown that, depending on the parameters of the game, there exist three different types of solutions for this class of games.

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Introduction

In a recent paper [Ref. 1] Simaan and Cruz have obtained the open-loop Stackelberg solution for a class of deterministic nonzero-sum two-person games under the leadership of one of the players. One of the properties of the Stackelberg strategies, as discussed in Ref. 1, is that if one of the players acts as the leader in the game then both players might benefit from this leadership in the sense that both of them might end up with better payoffs than the ones obtained from the Nash strategies.⁺ This property of the Stackelberg solution brings up the question of as how to decide on which player should lead the game and which player should follow. Further, there is the question of whether it is always profitable for either player to act as the leader rather than be the follower.

In this note, we address ourselves to these questions via a scalar nonzero-sum differential game problem which is related to Example 5 of Ref. 1. We show that there are situations in which a player would prefer to be the follower rather than be the leader and that this leads, in general, to three different types of solutions for this class of games. The concept introduced and the conditions derived for the scalar example can readily be extended to encompass a more general class of nonzero-sum game problems.

⁺It is important to note that we are restricting ourselves to noncooperative solution concepts. Otherwise, pareto-optimal solution should be considered in making the comparison.

A Linear Quadratic Differential Game

Consider the following generalized form of Example 5 of Ref. 1. The dynamics are described by the scalar linear differential equation

$$\dot{x} = u_1 - u_2, \quad x(0) = x_0, \quad (1)$$

where u_1 and u_2 are controlled by player 1 and player 2 respectively and are measurable functions of $t \in (0, 1]$ and x_0 . The costs to players 1 and 2 are given by the quadratic payoff functions J_1 and J_2 respectively, where

$$J_1(u_1, u_2) = \frac{1}{2} c_1 x_f^2 + \frac{1}{2c_p} \int_0^1 u_1^2 dt, \quad (2a)$$

$$J_2(u_1, u_2) = \frac{1}{2} c_2 x_f^2 + \frac{1}{2c_e} \int_0^1 u_2^2 dt, \quad (2b)$$

$c_p > 0$, $c_e > 0$, $c_1 \neq 0$, $c_2 \neq 0$, and x_f denotes the terminal state (i. e. $x(1)$). Note that this formulation becomes identical to Example 5 of Ref. 1 when $c_1 = 1$, $c_2 = -1$.

Now denote the Stackelberg control of the i 'th player when the j 'th player is the leader by u_{isj} and the corresponding Stackelberg payoff $J_i(u_{isj}, u_{2sj})$ by J_i^j . Then it follows from Eqs. (48) - (53) of Ref. 1 that the open-loop Stackelberg solution with player 2 as the leader is given by

$$u_{1s2} = - \frac{c_1 c_p (1 + c_1 c_p)}{(1 + c_1 c_p)^2 + c_2 c_e} x_0 \quad (3a)$$

$$u_{2s2} = \frac{c_e c_2}{(1+c_1 c_p)^2 + c_2 c_e} x_0, \quad (3b)$$

under the conditions

$$(1 + c_1 c_p) > 0 \quad (4a)$$

$$(1 + c_1 c_p)^2 + c_2 c_e > 0, \quad (4b)$$

and the corresponding Stackelberg payoffs are

$$J_1^2 = \frac{1}{2} \frac{c_1 (1+c_1 c_p)^3}{[(1+c_1 c_p)^2 + c_2 c_e]^2} x_0^2 \quad (5a)$$

$$J_2^2 = \frac{1}{2} \frac{c_2}{[(1+c_1 c_p)^2 + c_2 c_e]} x_0^2 \quad (5b)$$

Using a symmetry property of the original differential game, the open-loop Stackelberg solution with player 1 as the leader can readily be obtained from (3a) - (5b), i. e.

$$u_{1s1} = - \frac{c_1 c_p}{(1+c_2 c_e)^2 + c_1 c_p} x_0 \quad (6a)$$

$$u_{2s1} = \frac{c_2 c_e (1 + c_2 c_e)}{(1+c_2 c_e)^2 + c_1 c_p} x_0, \quad (6b)$$

under the conditions

$$(1 + c_2 c_e) > 0 \quad (7a)$$

$$(1 + c_e c_2)^2 + c_1 c_p > 0 \quad , \quad (7b)$$

and the corresponding Stackelberg payoffs are

$$J_1^1 = \frac{1}{2} \frac{c_1}{[(1+c_2 c_e)^2 + c_1 c_p]} x_0^2 \quad (8a)$$

$$J_2^1 = \frac{1}{2} \frac{c_2(1+c_2 c_e)^3}{[(1+c_2 c_e)^2 + c_1 c_p]^2} x_0^2 \quad (8b)$$

Denoting the Nash payoffs of players 1 and 2 by J_1^N and J_2^N respectively, it is certainly true that

$$J_1^1 < J_1^N \quad , \quad J_2^2 < J_2^N \quad , \quad (9)$$

that is, the leader will always do better (in the sense of achieving a lower payoff) than his Nash solution. [The inequalities in (9) are strict because of the assumption $c_1 \neq 0$, $c_2 \neq 0$.] However, relation (9) does not necessarily imply that the best each player can do (in a noncooperative sense) is to be the leader in the game. In fact, such a statement will not always be true as will be shown in the sequel.

In order to derive the conditions under which $J_1^1 \leq J_1^2$ and/or $J_2^2 \leq J_2^1$, we will first have to require relations (4a), (4b), (7a) and (7b) to be satisfied. Then $J_1^1 \leq J_1^2$ implies (after some straightforward but extensive manipulations) either

$$(i) \quad c_2 > 0 \quad (10a)$$

or

$$(ii) \quad c_2 < 0, \quad -\frac{2}{c_2 c_e} \leq 1 + \frac{2+c_1 c_p}{(1+c_1 c_p)^2} \quad (10b)$$

That is, if either (10a) or (10b) is satisfied, player 1 can achieve the lowest possible payoff (in a noncooperative sense and assuming that player 2 acts rationally) by being the leader in the game. [Note that if $J_i^1 = J_i^j$, $i \neq j$, then player i achieves the same payoff by being either the leader or the follower. In such a paradoxical situation we assume that he acts as a leader.]

Similarly, the conditions under which player 2 would rather prefer to be the leader (i. e. $J_2^2 \leq J_2^1$) are either

$$(iii) \quad c_1 > 0 \quad (11a)$$

or

$$(iv) \quad c_1 < 0, \quad -\frac{2}{c_1 c_p} \leq 1 + \frac{2+c_2 c_e}{(1+c_2 c_e)^2} \quad (11b)$$

provided that relations (4a), (4b), (7a) and (7b) are also satisfied.

To summarize these results and to indicate their immediate implications in a compact form, denote the set of $c_1 \neq 0$, $c_2 \neq 0$, $c_p > 0$, $c_e > 0$ which satisfy (4a), (4b), (7a) and (7b) by Ω . Further, denote the quadruple $\{c_1, c_2, c_p, c_e\}$ by a . Let Γ_1 be the set of $a \in \Omega$ which satisfy either (10a) or (10b), and Γ_2 be the set of $a \in \Omega$ which satisfy either (11a) or (11b). Then we have the following conclusions. (Note that any given a specifies the game completely).

$$(1) \quad \Gamma_1 \subset \Omega, \quad \Gamma_2 \subset \Omega$$

- (2) $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ (this will be proven in the sequel via a numerical example - Example 2)
- (3) $(\Omega - \Gamma_1) \cap (\Omega - \Gamma_2) \neq \emptyset$ (this will also be proven in the sequel via a numerical example - Example 4)
- (4) Player i wants to be the leader iff $\alpha \in \Gamma_i$.
- (5) Player i wants to be the follower iff $\alpha \in \Omega - \Gamma_i$.
- (6) Both players want player i to be the leader iff $\alpha \in \Gamma_i \cap (\Omega - \Gamma_j)$, $i \neq j$.
- (7) Either player wants himself to be the leader iff $\alpha \in \Gamma_1 \cap \Gamma_2$.
- (8) Either player wants himself to be the follower iff $\alpha \in (\Omega - \Gamma_1) \cap (\Omega - \Gamma_2)$.

Hence, associated with the nonzero-sum differential game considered in this note, we have 3 different types of solutions, depending on the parameters defining the game:

TYPE A: - CONCURRENT SOLUTION

If $\alpha \in \Gamma_i \cap (\Omega - \Gamma_j)$, $i \neq j$, it follows from item (6) that the players mutually benefit from the leadership of the i 'th player and hence they "collectively" decide to play the game under player i 'th leadership (even though it is a noncooperative game). We call this a concurrent solution, since there is no reason for either player to deviate from the corresponding Stackelberg solution which was computed under mutual agreement.

TYPE B: - NONCONCURRENT SOLUTION

If $\alpha \in \Gamma_1 \cap \Gamma_2$, either player knows that he will do best (in the noncooperative sense) if he himself is the leader (item 7). Hence either player will try to announce his strategy first and thus force the other

player to pick the Stackelberg strategy under his leadership. In this case, the one who can process his data faster will certainly be the leader and announce his policy first. However, if the "slower" player does not actually know that the other player can process his data faster than he does and/or if there is a delay in the information exchange between the two players (which is the case in many economic situations), then he might tend to announce a Stackelberg strategy under his own leadership quite unaware of the announcement of the other player; this certainly results in a nonequilibrium situation.

TYPE C: - STALEMATE SOLUTION

If $\alpha \in (\Omega - \Gamma_1) \cap (\Omega - \Gamma_2)$, then neither player wants to be the leader (item 8). Both players will rather prefer to wait for the opponent to announce his policy first -- which will result in a "stalemate". In order to come up with a reasonable solution for this case, one has to introduce some negotiation or bargaining between the players. The question of the existence and nature of the bargaining procedure that would result in a concurrent solution is yet an open problem that requires further investigation.

We next consider numerical examples to illustrate these three different types of solutions.

Example 1

$$\alpha = \{c_1 = 1, c_2 = -1, c_p = 1, c_e = 0.5\}$$

It can readily be checked that $\alpha \in \Gamma_2 \cap (\Omega - \Gamma_1)$, and hence this example admits a Type A solution with player 2 being the leader. This example can also be considered as a velocity-controlled pursuit-evasion game of

the nonzero-sum variety in which the pursuer (player 1) has less weight on his control than the evader (player 2) [i. e. $\frac{1}{c_p} < \frac{1}{c_e}$]. Under this condition the pursuer would prefer to wait and act second, and the evader would rather prefer to act first.

Different Stackelberg payoffs for this example are

$$J_1^1 = 0.400 x_0^2 \quad J_2^1 = -0.040 x_0^2$$

$$J_1^2 = 0.326 x_0^2 \quad J_2^2 = -0.143 x_0^2$$

Example 2

$$a = \{c_1 = 1, c_2 = -1, c_p = 0.2, c_e = 0.8\}$$

For this game $a \in \Gamma_1 \cap \Gamma_2$ and hence it admits a Type B solution, i. e. neither player 1 (the pursuer) nor player 2 (the evader) want to be the follower. Note that the only difference of this example from the previous one is that now the pursuer has more weight on his control than the evader.

Different Stackelberg payoffs for example 2 are

$$J_1^1 = 2.08 x_0^2 \quad J_2^1 = -0.069 x_0^2$$

$$J_1^2 = 2.11 x_0^2 \quad J_2^2 = -0.78 x_0^2$$

Example 3

$$a = \{c_1 = 1, c_2 = 1, c_p = 1, c_e = 0.5\}$$

This example also admits a Type B solution, i. e. $a \in \Gamma_1 \cap \Gamma_2$. Different Stackelberg payoffs for this game are

$$J_1^1 = 0.154 x_0^2 \quad J_2^1 = 0.160 x_0^2$$

$$J_1^2 = 0.198 x_0^2 \quad J_2^2 = 0.111 x_0^2$$

Example 4

$$\alpha = \{c_1 = -1, c_2 = -1, c_p = \frac{1}{3}, c_e = \frac{1}{3}\}$$

For this final example, $\alpha \in (\Omega - \Gamma_1) \cap (\Omega - \Gamma_2)$ and hence it admits a Type C solution. Both players want to be the follower and this leads to a "stalemate" solution. One has to introduce some cooperation between the players in order to derive a concurrent solution (if such a solution exists). Different Stackelberg payoffs for this example are

$$J_1^1 = -4.5 x_0^2 \quad J_2^1 = -12 x_0^2$$

$$J_1^2 = -12 x_0^2 \quad J_2^2 = -4.5 x_0^2$$

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References

1. Simaan, M., and Cruz, J. B., Jr., "On the Stackelberg Strategy in Nonzero-Sum Games," to appear in the Journal of Optimization Theory and Applications.